Listy Biometryczne – Biometrical Letters
Vol. 31 (1994), No. 1, 27-33

Some characterizations of a class of balanced two-way elimination of heterogeneity designs

Idzi Siatkowski

Department of Mathematical and Statistical Methods, Agricultural University of Poznań, Wojska Polskiego 28, 60-637 Poznań, Poland

Summary

We characterize the variance balance and efficiency balance of a class of two-way elimination of heterogeneity designs in relation to the property of equal treatment replications and in relation to the corresponding balance properties of the treatment-row and treatment-column subdesigns.

1. Introduction and preliminaries

Consider a *two-way elimination of heterogeneity design* in which v treatments are allocated on n experimental units arranged in b_1 rows and b_2 columns. Let $\mathbf{r} = (r_1,\ldots,r_v)'$, $\mathbf{k}_1 = (k_{11},\ldots,k_{1b_1})'$, and $\mathbf{k}_2 = (k_{21},\ldots,k_{2b_2})'$ denote the vector of treatment replications, the vector of row sizes, and the vector of column sizes, respectively, and let $\mathbf{R}$, $\mathbf{K}_1$, and $\mathbf{K}_2$ be the diagonal matrices with the successive elements of $\mathbf{r}$, $\mathbf{k}_1$, and $\mathbf{k}_2$ on their diagonals. Moreover, let $\mathbf{N}_1$ be the $v\times b_1$ treatment-row incidence matrix, let $\mathbf{N}_2$ be the $v\times b_2$ treatment-column incidence matrix, and let $\mathbf{N}_{12}$ be the $b_1\times b_2$ row-column incidence matrix.

A crucial role in the analysis of this design is played by the *C-matrix* defined by

$$\mathbf{C} = \mathbf{R} - \mathbf{N}_1\mathbf{K}_1^{-1}\mathbf{N}_1' - (\mathbf{N}_2 - \mathbf{N}_1\mathbf{K}_1^{-1}\mathbf{N}_{12})(\mathbf{K}_2 - \mathbf{N}_{12}'\mathbf{K}_1^{-1}\mathbf{N}_{12})^{-}(\mathbf{N}_2 - \mathbf{N}_1\mathbf{K}_1^{-1}\mathbf{N}_{12})'$$

$$= \mathbf{R} - \mathbf{N}_2\mathbf{K}_2^{-1}\mathbf{N}_2' - (\mathbf{N}_1 - \mathbf{N}_2\mathbf{K}_2^{-1}\mathbf{N}_{12}')(\mathbf{K}_1 - \mathbf{N}_{12}\mathbf{K}_2^{-1}\mathbf{N}_{12}')^{-}(\mathbf{N}_1 - \mathbf{N}_2\mathbf{K}_2^{-1}\mathbf{N}_{12}')' \quad ,$$

Key words: two-way elimination of heterogeneity design, row-column designs, efficiency-balance, variance-balance

where the minus superscript denotes any generalized inverse of a matrix; cf. Raghavarao and Federer (1975). The C-matrices of the two related subdesigns are

$$\mathbf{C}_h = \mathbf{R} - \mathbf{N}_h\mathbf{K}_h^{-1}\mathbf{N}_h'$$

with h = 1 for the treatment-row subdesign and h = 2 for the treatment-column subdesign, while the matrix $\mathbf{C}_0$ is defined as

$$\mathbf{C}_0 = \mathbf{R} - \mathbf{r}\mathbf{r}'/n \quad . \tag{1}$$

A two-way elimination of heterogeneity design is said to be *connected* if all treatment contrasts are unbiasedly estimable in the corresponding fixed linear model, for which it is necessary and sufficient that $r(\mathbf{C}) = v - 1$, where r(C) denotes the rank of **C**. A two-way elimination of heterogeneity design is said to be *variance-balanced* if every normalized estimable treatment contrast is estimated in the corresponding linear model with the same variance. It is known [cf. Kshirsagar (1957) and Singh, Dey, and Nigam (1979)] that a connected two-way elimination of heterogeneity design is variance-balanced if and only if

$$\mathbf{C} = \alpha\mathbf{Q}_v \quad \text{for some} \quad \alpha > 0 \ , \tag{2}$$

where $\mathbf{Q}_v = \mathbf{I}_v - \mathbf{1}_v\mathbf{1}_v'/v$ is the orthogonal projector on the ortho-complement of the subspace spanned by $\mathbf{1}_v$, the $v\times 1$ vector of ones. A two-way elimination of heterogeneity design is said to be *efficiency-balanced* if every estimable treatment contrast is estimated in the corresponding linear model with the same efficiency. Following Williams (1975), we observe that a connected two-way elimination of heterogeneity design is efficiency-balanced if and only if

$$\mathbf{C} = \theta\mathbf{C}_0 \quad \text{for some} \quad \theta \in (0,1] \ , \tag{3}$$

where $\mathbf{C}_0$ is as defined in (1). The balance properties for the treatment-row and treatment-column subdesigns are defined analogously, and the criteria (2) and (3) modify to

$$\mathbf{C}_h = \alpha_h\mathbf{Q}_v \quad \text{for some} \quad \alpha_h > 0 \tag{4}$$

and

$$\mathbf{C}_h = \theta_h\mathbf{C}_0 \quad \text{for some} \quad \theta_h \in (0,1) \ , \tag{5}$$

respectively, where h = 1, 2.

For all connected two-way elimination of heterogeneity designs with equal row sizes, equal column sizes, and whose C matrices can be represented in the form

$$\mathbf{C} = \mathbf{R} - \nu_1\mathbf{N}_1\mathbf{N}_1' - \nu_2\mathbf{N}_2\mathbf{N}_2' + \rho\mathbf{r}\mathbf{r}' \quad \text{for some} \quad \nu_1, \nu_2, \rho > 0 \ , \tag{6}$$

some new results are derived concerning characterizations of the balance properties of a design in relation to the property of equal treatment replications and in relation to the corresponding balance properties of the treatment-row and treatment-column subdesigns.

It is known, that (6) may hold also for designs with equal row sizes and equal column sizes, which do not satisfy the condition

$$\mathbf{C} = \mathbf{C}_1 + \mathbf{C}_2 - \mathbf{C}_0$$

considered by Baksalary and Shah (1990). An example, with 7 treatments allocated in 7 rows and 7 columns, is given by

$$\begin{matrix} * & 3 & 5 & * & 2 & * & * \\ * & * & 4 & 6 & * & 3 & * \\ * & * & * & 5 & 7 & * & 4 \\ 5 & * & * & * & 6 & 1 & * \\ * & 6 & * & * & * & 7 & 2 \\ 3 & * & 7 & * & * & * & 1 \\ 2 & 4 & * & 1 & * & * & * \end{matrix} \ ,$$

where the integers 1 through 7 denote distinct treatments and the asterisks indicate blanks; cf. Agrawal (1966). In this case $\mathbf{C}_1 = \mathbf{C}_2 = (7/3)\mathbf{Q}_7$ and $\mathbf{C}_0 = 3\mathbf{Q}_7$, thus yielding $\mathbf{C}_1 + \mathbf{C}_2 - \mathbf{C}_0 = (5/3)\mathbf{Q}_7$, whereas $\mathbf{C} = \mathbf{Q}_7$. But matrix $\mathbf{C}$ may be represented as in (6) with ρ=2/21 and any ν_1 and ν_2 such that $\nu_1 + \nu_2 = 1$.

2. Properties of balanced designs

Singh, Dey, and Nigam (1979) gave the following characterization of balance of two-way elimination of heterogeneity designs.

Theorem 1. For a connected two-way elimination of heterogeneity design with the number of treatments $v \geq 3$, any two of the following properties imply the third property:

(a) the design is efficiency-balanced,
(b) the design is variance-balanced,
(c) the design is equireplicated.

One of the topics of natural interest in the theory of two-way elimination of heterogeneity designs is investigating relationships between properties of a given design and the corresponding properties of its treatment-row and treatment-column subdesigns. If the property in question is the efficiency balance, then an immediate consequence of the criteria (3) and (5) is the following.

Theorem 2. For a connected two-way elimination of heterogeneity design with equal row sizes, equal column sizes, and such that $\mathbf{C} = \mathbf{R} - \nu_1\mathbf{N}_1\mathbf{N}_1' - \nu_2\mathbf{N}_2\mathbf{N}_2' + \rho\mathbf{rr}'$ for some $\nu_1,\ \nu_2,\ \rho > 0$, any two of the following properties imply the third property:

(a) the design is efficiency-balanced,
(b) the treatment-row subdesign is efficiency-balanced,
(c) the treatment-column subdesign is efficiency-balanced.

Proof. If the row sizes of the design are all equal to k_1 and the column sizes are all equal to k_2, then condition (6), which may be reexpressed as

$$\mathbf{C} = k_1\nu_1(\mathbf{R} - \mathbf{N}_1\mathbf{N}_1'/k_1) + k_2\nu_2(\mathbf{R} - \mathbf{N}_2\mathbf{N}_2'/k_2) - n\rho(\mathbf{R} - \mathbf{rr}'/n)\ ,$$

is equivalent to

$$\mathbf{C} = k_1\nu_1\mathbf{C}_1 + k_2\nu_2\mathbf{C}_2 - n\rho\mathbf{C}_0\ . \tag{7}$$

In view of (7), the theorem is an immediate consequence of criteria (3) and (5).

□

Similarly as for efficiency-balanced designs, relationships between variance properties of a design and its subdesigns may be formulated.

Theorem 3. For a connected two-way elimination of heterogeneity design with the number of treatments $v \geq 3$, equal row sizes, equal column sizes, and such that $\mathbf{C} = \mathbf{R} - \nu_1\mathbf{N}_1\mathbf{N}_1' - \nu_2\mathbf{N}_2\mathbf{N}_2' + \rho\mathbf{rr}'$ for some $\nu_1, \nu_2,\ \rho > 0$, any three of the following properties imply the fourth property:

(a) the design is variance-balanced,
(b) the treatment-row subdesign is variance-balanced,
(c) the treatment-column subdesign is variance-balanced,
(d) the design is equireplicated.

Proof. In view of (7), the theorem is an immediate consequence of criteria (2) and (4) and the fact that if $v \geq 3$, then $\mathbf{C}_0$ is proportional to $\mathbf{Q}_v$ if and only if the design is equireplicated.

□

It turns out that for the subclass of all two-way elimination of heterogeneity designs whose C-matrices have the representation (6), the part "(a) and (b) ⇒ (c)" of Theorem 1 may be substantially strengthened.

Theorem 4. For a connected two-way elimination of heterogeneity design with equal row sizes and column sizes, with the property that each of $v \geq 3$ treatment occurs in each row as well as in each column at most once, and with the C-matrix of the form $\mathbf{C} = \mathbf{R} - \nu_1\mathbf{N}_1\mathbf{N}_1' - \nu_2\mathbf{N}_2\mathbf{N}_2' + \rho\mathbf{rr}'$ for some $\nu_1, \nu_2,\ \rho > 0$, consider the following statements:

(a) the design is efficiency-balanced,
(b) the design is variance-balanced,
(c) the design is equireplicated.

Then (a) ⇔ (b) ∧ (c) and (b) ⇔ (a) ∧ (c).

Proof. The parts (b) ∧ (c) ⇒ (a) and (a) ∧ (c) ⇒ (b) are inherent in Theorem 1. If (a) holds, then combining (6) with (3) yields

$$\mathbf{R} - \nu_1\mathbf{N}_1\mathbf{N}_1' - \nu_2\mathbf{N}_2\mathbf{N}_2' + \rho\mathbf{r}\mathbf{r}' = \theta(\mathbf{R} - \mathbf{r}\mathbf{r}'/n) \ . \tag{8}$$

According to the assumption, the matrices $\mathbf{N}_1$ and $\mathbf{N}_2$ are binary, and therefore comparing the i-th diagonal elements on the two sides of (8) leads to the equality

$$r_i - \nu_1 r_i - \nu_2 r_i + \rho r_i^2 = \theta(r_i - r_i^2/n) \ .$$

Hence it follows that r_i is independent of i, thus implying (c). If (b) holds, then combining (6) with (2) yields

$$\mathbf{R} - \nu_1\mathbf{N}_1\mathbf{N}_1' - \nu_2\mathbf{N}_2\mathbf{N}_2' + \rho\mathbf{r}\mathbf{r}' = \alpha(\mathbf{I}_v - \mathbf{1}_v\mathbf{1}_v'/v) \ , \tag{9}$$

and comparing the i-th diagonal elements on the two sides of (9) leads to the equality

$$r_i - \nu_1 r_i - \nu_2 r_i + \rho r_i^2 = \alpha(1 - 1/v) \ . \tag{10}$$

Hence

$$(r_i - r_{i'})[1 - \nu_1 - \nu_2 + \rho(r_i + r_{i'})] = 0 \quad \text{for every} \quad i,i' = 1,\dots,v, \ i \neq i' \ . \tag{11}$$

It is clear that (10) also entails the inequality $1 - \nu_1 - \nu_2 + \rho r_i > 0$. Consequently, the expression in brackets in (11) is positive, which leads to (c).

❑

For designs with C-matrices of the form (6), relationships between the properties of efficiency balance and commutativity, defined by

$$\mathbf{N}_1\mathbf{K}_1^{-1}\mathbf{N}_1'\mathbf{R}^{-1}\mathbf{N}_2\mathbf{K}_2^{-1}\mathbf{N}_2' = \mathbf{N}_2\mathbf{K}_2^{-1}\mathbf{N}_2'\mathbf{R}^{-1}\mathbf{N}_1\mathbf{K}_1^{-1}\mathbf{N}_1' \ ,$$

or, equivalently

$$\mathbf{A}_1\mathbf{A}_2 = \mathbf{A}_2\mathbf{A}_1 \ ,$$

where

$$\mathbf{A}_h = \mathbf{R}^{-1/2}\mathbf{C}_h\mathbf{R}^{-1/2} \ , \ h = 0, 1, 2 \ ,$$

is given in Theorem 5 below.

Theorem 5. A connected two-way elimination of heterogeneity design with equal row sizes, equal column sizes, and such that $\mathbf{C} = \mathbf{R} - \nu_1\mathbf{N}_1\mathbf{N}_1' - \nu_2\mathbf{N}_2\mathbf{N}_2' + \rho\mathbf{rr}'$ for some ν_1, ν_2, $\rho > 0$ is efficiency-balanced if and only if it satisfies the commutativity property $\mathbf{A}_1\mathbf{A}_2 = \mathbf{A}_2\mathbf{A}_1$ and the value of $\nu_1 k_1 \varphi_{1i} + \nu_2 k_2 \varphi_{2i}$ is the same for $i = 1,\dots,v-1$, where the nonzero eigenvalues $\varphi_{h1},\dots, \varphi_{h,v-1}$, $h = 1, 2$, are ordered correspondingly to a fixed set of common eigenvectors of $\mathbf{A}_1$ and $\mathbf{A}_2$.

Proof. It is clear that (7) may be reformulated as

$$\mathbf{A} = \nu_1 k_1 \mathbf{A}_1 + \nu_2 k_2 \mathbf{A}_2 - \rho n \mathbf{A}_0 \ , \tag{12}$$

where

$$\mathbf{A} = \mathbf{R}^{-1/2}\mathbf{C}\mathbf{R}^{-1/2}.$$

Postmultiplying in (6) by $\mathbf{1}_v$ leads to the equality $\mathbf{0} = (1 - \nu_1 k_1 - \nu_2 k_2 + \rho n)\mathbf{r}$, and hence

$$\rho n = \nu_1 k_1 + \nu_2 k_2 - 1 \ . \tag{13}$$

Substituting (13) into (12) yields

$$\mathbf{A}_0 - \mathbf{A} = \nu_1 k_1(\mathbf{A}_0 - \mathbf{A}_1) + \nu_2 k_2(\mathbf{A}_0 - \mathbf{A}_2) \ ,$$

which implies that, for every $i = 1,\dots,v-1$,

$$1 - \varphi_i = \nu_1 k_1(1 - \varphi_{1i}) + \nu_2 k_2(1 - \varphi_{2i}) \ . \tag{14}$$

In view of (12), if $\mathbf{A} = \varepsilon\mathbf{A}_0$, then $\nu_1 k_1 \mathbf{A}_1 + \nu_2 k_2 \mathbf{A}_2 = (\varepsilon + \rho n)\mathbf{A}_0$, which clearly implies that $\mathbf{A}_1\mathbf{A}_2 = \mathbf{A}_2\mathbf{A}_1$ and $\nu_1 k_1 \varphi_{1i} + \nu_2 k_2 \varphi_{2i} = \varepsilon + n\rho$ for $i = 1,2,\dots,v-1$. Conversely, if $\mathbf{A}_1\mathbf{A}_2 = \mathbf{A}_2\mathbf{A}_1$ and $\nu_1 k_1 \varphi_{1i} + \nu_2 k_2 \varphi_{2i}$ is constant, then (14) shows that the eigenvalues $\varphi_1,\dots,\varphi_{v-1}$ of $\mathbf{A}$ are all equal, i.e. $\mathbf{A}$ is a scalar multiple of $\mathbf{A}_0$.

❑

Acknowledgements

I am grateful to Professor Jerzy K. Baksalary for helpful discussions.

REFERENCES

Agrawal, H. L. (1966). Some systematic methods of construction of designs for two-way elimination of heterogeneity. *Calcutta Statist. Assoc. Bull.* **15**, 93-108.

Baksalary, J. K. and K. R. Shah (1992). Some properties of two-way elimination of heterogeneity designs. *Probability, Statistics and Design of Experiments* (R. R. Bahadur, ed.), Wiley Eastern, Delhi.

Kshirsagar, A. M. (1957). On balancing in designs in which heterogeneity is eliminated in two directions. *Calcutta Statist. Assoc. Bull.* **7**, 161-166.

Puri, P. D. and A. K. Nigam (1975). On patterns of efficiency-balanced designs. *J. Roy. Statist. Soc. Ser. B* **37**, 457-458.

Raghavarao, D. and W. T. Federer (1975). On connectedness in two-way elimination of heterogeneity designs. *Ann. Statist.* **3**, 730-735.

Singh, M., A. Dey, and A. K. Nigam (1979). Two-way elimination of heterogeneity - II. *Sankhya Ser. B* **40**, 227-235.

Williams, E. R. (1975). Efficiency-balanced designs. *Biometrika* **62**, 686-689.

Received 15 August 1994